

# STABILITY OF THE RECOVERY OF MISSING SAMPLES IN DERIVATIVE OVERSAMPLING

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**ABSTRACT.** This paper deals with the problem of reconstructing a band-limited signal when a finite subset of its samples and of its derivative are missing. The technique used, due to P.J.S.G. Ferreira, is based on the use of a particular frame for band-limited functions and the relative oversampling formulas. We study the eigenvalues of the matrices arising in the procedure of recovering the lost samples, finding estimates of their eigenvalues and their dependence on the oversampling parameter and on the number of missing samples. When the missing samples are consecutive, the problem may become very ill-conditioned. We present a numerical procedure to overcome this difficulty, also in presence of noisy data, by using Tikhonov regularization techniques.

## 1. INTRODUCTION

This paper deals with the problem of recovering missing samples of band-limited functions and of its derivative via frame reconstruction sampling formulas. A band-limited signal is a function which belongs to the space  $B_\omega$  of functions in  $L^2(\mathbb{R})$  whose Fourier transforms have support in  $[-\omega, \omega]$ . Functions in this space can be expanded in terms of the orthonormal basis of translates of the *sinc* function. The coefficients of the expansion are the samples of the function at a uniform grid on  $\mathbb{R}$ , with “density”  $\omega/\pi$  (*Nyquist* density). Such expansions, called sampling formulas, have been generalized by replacing the orthonormal basis with more general families, like Riesz bases and frames, formed by the translates of one or more functions (*generators*). Frames, unlike Riesz basis, are overcomplete and their redundancy provides a perfect tool for the recovery of missing data.

The problem of recovering missing samples has been investigated first by P.J.S.G. Ferreira in [F], where it is shown that, in the case of a generalized Kramer sampling, under suitable oversampling assumptions, any finite set of missing samples can be recovered from the others by solving a linear system  $(I - S)x = b$ , where the matrix  $S$  is positive definite. Moreover the author studies the eigenvalues of the matrix  $S$  in dependence of the oversampling parameter and the number of missing samples. Successively D.M.S. Santos and Ferreira considered the case of a two-channel derivative oversampling formula obtained by projecting the generators of a Riesz basis of the space  $B_\omega$  and their duals into the space  $B_{\omega_a}$  with  $\omega_a < \omega$  [SF]. With this technique they obtain a pair of dual frames, although the dual frame is not the canonical one. In their paper the authors show that a finite number of missing samples either of the function or of its derivative can be recovered, solving in each case a non singular linear system similar to the one-channel case. However the authors do not consider the case when samples both of the function and the

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*Key words and phrases.* frame, Riesz basis, shift-invariant space, sampling formulas, band-limited functions.

derivative are simultaneously missing.

The technique of Santos and Ferreira has been applied to more general two-channels by J. M. Kim and K. H. Kwon [KK], who gave sufficient conditions for the recovery of missing samples, from a single channel and from both channels. However, as the authors observe, these conditions do not apply to the derivative sampling formulas studied in [SF].

Successively, in [DP], the author gave new derivative oversampling formulas of any order, where the dual generator is the canonical dual and in [DP1] found an expression of the dual in the two-channel case. The use of the canonical dual allowed the author to prove in [DP1] that simultaneously missing samples of the function and of the derivative can be recovered. However the generators of the canonical duals are much less explicit than the non-canonical dual used by Santos and Ferreira see [DP1, (5.2) p. 178] and (2.10).

In this paper we study the problem of the recovery of missing samples of the function and of its derivative in Ferreira's two-channel derivative formula. The technique we use, first proposed in [SF], extends in a natural way that for one channel in [F] and consists in solving a system  $(I - S)X = C$ , where the matrix  $S$  (see (3.8)) is a block matrix depending on the dual generators and on the position of the missing samples, and the unknowns  $X$  are the missing samples of the function and its derivative. To the best of our knowledge, no results are known about the possibility of solving this system, i.e. of recovering simultaneous missing samples of the function and of its derivative.

We obtain estimates of the minimum and maximum eigenvalues of the block submatrices of the matrix  $S$ , show that in some cases the matrix reduces to a lower triangular matrix and provide several numerical experiments on the dependence of the eigenvalues on the parameters of the problem, like the oversampling parameter  $r$  and the number of missing samples. Moreover, we experiment the recovery of missing samples and the reconstruction of a signal, paying particular attention to the computational aspects, analyzing the ill-conditioned problem of contiguous samples, also with noisy data.

The paper is organized as follows. In Section 2 we establish notation and collect some results to be used later. In Section 3 we describe the technique for the recovery of the missing samples. Section 4 is dedicated to the study of the eigenvalues of the coefficient matrices in the system, mostly when the missing samples are equidistant. In this case, we find estimates of the minimum and maximum eigenvalues of the diagonal submatrices  $S_{11}$  and  $S_{22}$  of  $S$  (see (3.8)). Moreover, we find that in some cases the matrix  $S$  is lower triangular and that half of its eigenvalues are equal to  $2r - r^2$ , half to  $r^2$ , where  $r$  is the oversampling parameter. This extends to two channels a result in [F]. We also present some numerical experiments supporting the theory. Finally, in Section 5, we analyze the case of contiguous missing samples both for one and two channels, when the problem may become very ill-conditioned. We present a numerical procedure to solve it, also in presence of noisy data, via regularization techniques.

## 2. PRELIMINARIES

In this section we collect some results on frames to be used later. We begin by introducing some notation. The Fourier transform of a function  $f$  in  $L^1(\mathbb{R})$  is

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-it\xi} dt.$$

In this paper vectors in  $\mathbb{C}^N$  are to be considered as column-vectors, however, to save space, we shall write  $x = (x_1, x_2, \dots, x_N)$  to denote the column-vector whose components are  $x_1, \dots, x_N$ . Let  $t_o$  be a positive number; we shall say that a subspace  $H$  of  $L^2(\mathbb{R})$  is  $t_o$ -shift-invariant if  $H$  is invariant under translation by  $t_o$ . Given a subset  $\Phi = \{\varphi_j, j = 1, 2\}$  of  $H$ , we denote by  $E_{\Phi, t_o}$  the set

$$E_{\Phi, t_o} = \{\tau_{nt_o} \varphi_j \mid n \in \mathbb{Z}, j = 1, 2\},$$

where  $\tau_a f(x) = f(x + a)$ . The family  $E_{\Phi, t_o}$  is a frame for  $H$  if and only if there exist two constants  $0 < A \leq B$  such that  $A\|f\|^2 \leq \sum_{j=1}^2 \sum_{n \in \mathbb{Z}} |\langle f, \tau_{nt_o} \varphi_j \rangle|^2 \leq B\|f\|^2$  for all  $f \in H$ ; the constants  $A$  and  $B$  are called frame bounds. Denote by  $T_{\Phi, t_o}^* : H \rightarrow \ell^2(\mathbb{Z}; \mathbb{C}^N)$  the adjoint of  $T_{\Phi, t_o}$ , defined by  $(T_{\Phi, t_o}^* f)_j(n) = \langle f, \tau_{nt_o} \varphi_j \rangle$ ,  $n \in \mathbb{Z}, j = 1, 2$ . The operator  $T_{\Phi, t_o} T_{\Phi, t_o}^* : H \rightarrow H$  is called *frame operator*. Denote by  $\Phi^*$  the family  $\{\varphi_j^*, j = 1, 2\}$ , where

$$(2.1) \quad \varphi_j^* = (T_{\Phi, t_o} T_{\Phi, t_o}^*)^{-1} \varphi_j \quad j = 1, 2.$$

If  $E_{\Phi, t_o}$  is a frame for  $H$  then  $E_{\Phi^*, t_o}$  is also a frame, called the *canonical dual frame*, and  $T_{\Phi, t_o} T_{\Phi^*, t_o}^* = T_{\Phi^*, t_o} T_{\Phi, t_o}^* = I$ ; explicitly

$$(2.2) \quad f = \sum_{j=1}^2 \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j^* \rangle \tau_{nt_o} \varphi_j = \sum_{j=1}^2 \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j \rangle \tau_{nt_o} \varphi_j^* \quad \forall f \in H.$$

The elements of  $\Phi$  are called *generators* of the frame and the elements of  $\Phi^*$  *canonical dual generators*. Given a frame  $E_{\Phi, t_o}$  of a Hilbert space  $H$ , the expansion of an element in terms of the generators is not unique; a frame  $\{\tau_{kt_o} \varphi_1^d, \tau_{kt_o} \varphi_2^d, k \in \mathbb{Z}\}$  such that

$$(2.3) \quad f = \sum_{j=1}^2 \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} \varphi_j^d \rangle \tau_{nt_o} \varphi_j \quad \forall f \in H$$

is called *dual frame* for  $E_{\Phi, t_o}$ . We remind that the coefficients  $\{\langle f, \tau_{nt_o} \varphi_j^* \rangle, n, j \in \mathbb{Z}\}$  with respect to the *canonical dual frame* have the minimal  $\ell^2$  norm among all the sequences that represent an element  $f$  in terms of a given frame  $E_{\Phi, t_o}$ . If the family  $E_{\Phi, t_o}$  is a frame for  $H$  and the operator  $T_{\Phi, t_o}$  is injective, then  $E_{\Phi, t_o}$  is called a *Riesz basis*.

Let  $E_{\Phi, t_o}$  be a frame of a  $t_o$ -shift-invariant Hilbert space  $H$  with canonical dual generators  $\Phi^*$ . Denote by  $P$  the orthogonal projection on a  $t_o$ -shift-invariant subspace  $V$  of  $H$ , then  $E_{P\Phi, t_o}$  and  $E_{P\Phi^*, t_o}$  are families of dual frames for  $V$ , i.e.

$$f = \sum_{j=1}^2 \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} P \varphi_j^* \rangle \tau_{nt_o} P \varphi_j = \sum_{j=1}^2 \sum_{n \in \mathbb{Z}} \langle f, \tau_{nt_o} P \varphi_j \rangle \tau_{nt_o} P \varphi_j^* \quad \forall f \in V.$$

Note that  $E_{P\Phi^*, t_o}$  is not necessarily the canonical dual.

In this paper we study families  $E_{\Phi, t_o}$ ,  $\Phi = \{\varphi_1, \varphi_2\}$  where  $\varphi_1$  and  $\varphi_2$  have Fourier transform  $\widehat{\varphi_1}(x) = \chi_{[-\omega, \omega]}(x)$  and  $\widehat{\varphi_2}(x) = ix\chi_{[-\omega, \omega]}(x)$  and  $t_o$  and  $\omega$  are such that

$\omega \leq \frac{2\pi}{t_o} \leq 2\omega$ . The interest of these families in applications resides in their connection with derivative sampling formulas for the space  $B_\omega$ . To simplify notation, throughout the paper we shall set

$$h = \frac{2\pi}{t_o}.$$

The family  $E_{\Psi, t_o}$ ,  $\Psi = \{\psi_1, \psi_2\}$  defined by

$$(2.4) \quad \widehat{\psi}_1(x) = \chi_{[-h, h]}(x) \quad \widehat{\psi}_2(x) = ix\chi_{[-h, h]}(x)$$

is a Riesz basis for the space  $B_h$  (see [Hi, p.135]). The Fourier transforms of the dual generators are

$$(2.5) \quad \widehat{\psi}_1^*(x) = \frac{1}{h}(1 - \frac{|x|}{h})\chi_{[-h, h]}(x) \quad \widehat{\psi}_2^*(x) = \frac{i}{h^2} \text{sign}(x)\chi_{[-h, h]}(x),$$

hence the dual generators are

$$\psi_1^*(x) = \frac{1}{\sqrt{2\pi}} \text{sinc}^2\left(\frac{hx}{2}\right) \quad \psi_2^*(x) = -\frac{1}{\sqrt{2\pi}} x \text{sinc}^2\left(\frac{hx}{2}\right),$$

where  $\text{sinc}(x) = \sin(x)/x$ . For any  $f \in B_h$  the coefficients of the expansion in (2.2) are

$$\langle f, \tau_{-nt_o}\psi_1 \rangle = \sqrt{2\pi}f(nt_o) \quad \langle f, \tau_{-nt_o}\psi_2 \rangle = -\sqrt{2\pi}f'(nt_o) \quad \forall n \in \mathbb{Z}.$$

thus the expansion formula (2.2) becomes

$$(2.6) \quad f(x) = \sum_n \left( f(nt_o) \text{sinc}^2\left(\frac{h}{2}(x - nt_o)\right) + f'(nt_o) \text{sinc}^2\left(\frac{h}{2}(x - nt_o)\right) \right).$$

for any  $f \in B_h$ . Note that the convergence is uniform. This formula is called a *first order derivative sampling formula*. The sampling frequency for an  $N$ -channel formula is  $\omega/N\pi$  and is called *Nyquist frequency*. By using a frame instead of a Riesz basis, in [SF] the authors obtained a two-channel derivative formula where the sampling frequency in each channel is greater than the Nyquist frequency  $\omega/2\pi$  (*first order derivative oversampling formula*). The frame is obtained by projecting on  $B_\omega$  the Riesz basis  $E_{\Psi, t_o}$  for  $B_h$  where  $h = 2\pi/t_o$  and  $\omega < h$ . We denote by  $r$  the ratio

$$(2.7) \quad r = \frac{\omega}{h} = \frac{\omega t_o}{2\pi}.$$

Since the projection commutes with the translations, to project the family  $E_\Psi$  and the dual family, it is sufficient to project on  $B_\omega$  the generators and the dual generators. Thus, by (2.4), we obtain that the generators of the frame  $E_{\Phi, t_o}$  are defined by

$$(2.8) \quad \widehat{\varphi}_1(x) = \chi_{[-\omega, \omega]}(x) \quad \widehat{\varphi}_2(x) = ix\chi_{[-\omega, \omega]}(x)$$

and by (2.5) we obtain the Fourier transform of the dual generators

$$(2.9) \quad \widehat{\varphi}_1^*(x) = \frac{r}{\omega} \left(1 - \frac{r}{\omega}|x|\right) \chi_{[-\omega, \omega]}(x) \quad \widehat{\varphi}_2^*(x) = i \frac{r^2}{\omega^2} \text{sign}(x) \chi_{[-\omega, \omega]}(x).$$

A simple calculation shows that

$$\tilde{\varphi}_1(x) = \frac{1}{\sqrt{2\pi}} \left[ 2r(1-r)\text{sinc}(\omega x) + r^2 \text{sinc}^2\left(\frac{\omega x}{2}\right) \right] \quad (2.10)$$

$$\tilde{\varphi}_2(x) = -\frac{1}{\sqrt{2\pi}} x r^2 \text{sinc}^2\left(\frac{\omega x}{2}\right).$$

Thus we obtain the following derivative oversampling formula

$$(2.11) \quad f(x) = \sqrt{2\pi} \sum_{k \in \mathbb{Z}} \left( f(kt_o) \tau_{kt_o} \tilde{\varphi}_1 - f'(kt_o) \tau_{kt_o} \tilde{\varphi}_2 \right) \quad \forall f \in B_\omega$$

[SF]. We observe that  $1/r = 2\pi/\omega t_o$  is the ratio between the sampling frequency and the Nyquist frequency and that  $r \in (0, 1)$ . We shall be mainly interested to the case  $1/2 < r < 1$ , since for  $0 < r \leq 1/2$  it is possible to use one channel separately for the function and for its derivative [SF]. Note that if  $r$  is close to 1, the frame is close to a Riesz basis, while if  $r$  is small the frame is very redundant.

### 3. THE SYSTEM FOR THE RECOVERY OF MISSING SAMPLES

In this section we briefly describe the method for the recovery of a finite number of missing samples via two-channel derivative oversampling. We may rewrite equation (2.11) as

$$(3.1) \quad f(x) = \sqrt{2\pi} \sum_{i=1}^2 \sum_{k \in \mathbb{Z}} (-1)^{i-1} f^{(i-1)}(kt_o) \tilde{\varphi}_i(x - kt_o)$$

and computing the derivative of both sides, we obtain

$$(3.2) \quad f'(x) = \sqrt{2\pi} \sum_{i=1}^2 \sum_{k \in \mathbb{Z}} (-1)^{i-1} f^{(i-1)}(kt_o) \tilde{\varphi}_i'(x - kt_o).$$

Here

$$(3.3) \quad \tilde{\varphi}_1'(x) = \frac{2r}{\sqrt{2\pi}} \frac{1}{x} \left[ (1-r) (\cos(\omega x) - \text{sinc}(\omega x)) + \right. \\ \left. + r \text{sinc}\left(\frac{\omega x}{2}\right) \left( \cos\left(\frac{\omega x}{2}\right) - \text{sinc}\left(\frac{\omega x}{2}\right) \right) \right]$$

$$(3.4) \quad \tilde{\varphi}_2'(x) = -\frac{1}{\sqrt{2\pi}} r^2 \text{sinc}\left(\frac{\omega x}{2}\right) \left[ 2 \cos\left(\frac{\omega x}{2}\right) - \text{sinc}\left(\frac{\omega x}{2}\right) \right].$$

Observe that  $\tilde{\varphi}_2'(0) = 0$ .

Let  $\mathcal{U} = \{l_1, l_2, \dots, l_N\} \subset \mathbb{Z}$  and let  $\{f(l_j t_o), f'(l_j t_o) \mid 1 \leq j \leq N\}$  be the corresponding set of missing samples. By evaluating equations (3.1) and (3.2) in  $l_k t_o$  and separating the unknown samples from the known ones, we obtain

$$(3.5) \quad f(l_k t_o) - \sqrt{2\pi} \sum_{i=1}^2 \sum_{j=1}^N (-1)^{i-1} f^{(i-1)}(l_j t_o) \tilde{\varphi}_i((l_k - l_j) t_o) = c_k \\ f'(l_k t_o) - \sqrt{2\pi} \sum_{i=1}^2 \sum_{j=1}^N (-1)^{i-1} f^{(i-1)}(l_j t_o) \tilde{\varphi}_i'((l_k - l_j) t_o) = c_{k+N},$$

$1 \leq k \leq N$ , where

$$(3.6) \quad \begin{aligned} c_k &= \sqrt{2\pi} \sum_{i=1}^2 \sum_{n \in \mathbb{Z} \setminus \mathcal{U}} (-1)^{i-1} f^{(i-1)}(nt_o) \tilde{\varphi}_i(l_k t_o - nt_o) \\ c_{k+N} &= \sqrt{2\pi} \sum_{i=1}^2 \sum_{n \in \mathbb{Z} \setminus \mathcal{U}} (-1)^{i-1} f^{(i-1)}(nt_o) \tilde{\varphi}_i'(l_k t_o - nt_o). \end{aligned}$$

Equations (3.5) form a system of  $2N$  equations in the  $2N$  unknowns

$$(3.7) \quad f(l_1 t_o), \dots, f(l_N t_o), f'(l_1 t_o), \dots, f'(l_N t_o)$$

which can be written in matrix form: denote by  $S = S(\mathcal{U}, r)$  the real matrix

$$(3.8) \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

where  $S_{11}, S_{12}, S_{21}, S_{22}$  are the submatrices whose entries are

$$(3.9) \quad \begin{aligned} S_{11}(k, j) &= \sqrt{2\pi} \tilde{\varphi}_1((l_k - l_j)t_o) & S_{12}(k, j) &= -\sqrt{2\pi} \tilde{\varphi}_2((l_k - l_j)t_o) \\ S_{21}(k, j) &= \sqrt{2\pi} \tilde{\varphi}_1'((l_k - l_j)t_o) & S_{22}(k, j) &= -\sqrt{2\pi} \tilde{\varphi}_2'((l_k - l_j)t_o) \end{aligned}$$

$k, j = 1, \dots, N$ . Then the system (3.5) may be written

$$(3.10) \quad (I - S)Z = C$$

where  $I$  is the  $2N \times 2N$  identity matrix,  $C = (c_k)_{k=1}^{2N}$  is defined by (3.6) and  $Z = (X_1, X_2, \dots, X_N, Y_1, Y_2, \dots, Y_N)$ . The unknowns  $X_k$  are the missing samples of the function and the  $Y_k$  are the missing samples of the derivative.

Notice that, by (2.10), (3.3) and (3.4), if  $r = 1$  then  $S$  is the identity matrix. The four submatrices are real,  $S_{11}$  and  $S_{22}$  are symmetric, while  $S_{12}$  and  $S_{21}$  are antisymmetric. Moreover, if the distance between two consecutive  $l_k$  is constant, then the four submatrices are Toeplitz, but  $S$  is not Toeplitz.

Suppose that in (3.7) only the values  $\{X_i = f(l_i t_o), \quad i = 1, \dots, N\}$  are missing, while all the values of  $f'$  are known. By considering only the first  $N$  equations of the system (3.10) and by separating the known from the unknown samples, we obtain

$$(I - S_{11})X = C_1 + S_{12}Y,$$

where  $I$  is the  $N \times N$  identity matrix,  $X = (X_k)_{k=1}^N$ ,  $Y = (f'(l_k t_o))_{k=1}^N$  and  $C_1 = (c_k)_{k=1}^N$ . This equation may be rewritten

$$(3.11) \quad (I - S_{11})X = B_1$$

where  $B_1 = C_1 + S_{12}Y$ . Similarly, if in (3.7) only the values  $\{Y_i = f'(l_i t_o), \quad i = 1, \dots, N\}$  are missing, and the samples of  $f$  are known, one solves the system

$$(3.12) \quad (I - S_{22})Y = B_2$$

in the unknowns  $Y_i, i = 1, \dots, N$ , where  $C_2 = (c_k)_{k=1+N}^{2N}$  and  $B_2 = C_2 + S_{21}X$ .

If  $r \in (0, 1)$ , then the eigenvalues of the symmetric matrices  $S_{11}$  and  $S_{22}$  are in the interval  $(0, 1)$  [SF]. This implies that the recovery of a finite number of missing samples of the function is possible, if all the samples of the derivative are known and, vice-versa, samples of the derivative can be recovered, if all the samples of the function are known. In the next section we shall find bounds for the minimum and

maximum eigenvalues of the matrices  $S_{11}$  and  $S_{22}$ .

To the best of our knowledge no results are known about the possibility of solving system (3.10), i.e. of recovering simultaneous missing samples of the function and of its derivative. Notice that the system is solvable if and only if 1 is not an eigenvalue of the matrix  $S$ . On the basis of numerical evidence we conjecture that all its eigenvalues are real and that they lie in the interval  $(0, 1)$  for all  $r \in (0, 1)$ . In Section 4 we present several numerical experiments supporting our conjecture.

#### 4. THE STABILITY OF THE MATRICES.

This section is dedicated to the study of the eigenvalues of the matrices  $S$ ,  $S_{11}$  and  $S_{22}$  in the two-channel system (3.10). First, we briefly summarize the one-channel case and recall some stability results matrix obtained in [F]. The one-channel formula is

$$(4.1) \quad f(x) = \frac{\omega t_o}{\pi} \sum_{n \in \mathbb{Z}} f(nt_o) \frac{\sin \omega(x - nt_o)}{\omega(x - nt_o)} \quad \forall f \in B_\omega$$

where  $t_o < \frac{\pi}{\omega}$ . In this case the oversampling parameter is  $r = \omega t_o / \pi$ . Let  $U = \{l_1, l_1, \dots, l_N\}$  be a set of integer numbers and

$$(4.2) \quad \{f(l_k t_o), k = 1, \dots, N\}$$

the set of missing samples; then the system to be solved is

$$(4.3) \quad (I - R)X = B$$

where  $I$  is the identity  $N \times N$  matrix

$$(4.4) \quad R(j, k) = r \operatorname{sinc}(\pi r(l_j - l_k)), \quad j, k = 1, \dots, N,$$

$X = \{f(l_k t_o), k = 1, \dots, N\}$ ,  $B = (b_1, b_2, \dots, b_N)$ , and

$$(4.5) \quad b_j = r \sum_{k \notin U} f(kt_o) \operatorname{sinc}(\pi r(l_j - l_k)) \quad j = 1, \dots, N.$$

The matrix  $R$  is symmetric and positive definite. In [F] the author shows that all its eigenvalues are in  $(0, 1)$  and observes that there are two situations that lead to a ill-conditioned problem: when  $r$  is close to 1 and when the integers  $l_k$  are contiguous. In the first case, the frame is close to a Riesz basis, when the recovery is impossible and  $R$  becomes the identity matrix. In the second case the maximum eigenvalue of the matrix  $R$  grows rapidly with  $N$ ; moreover it can be close to 1 also for small values of  $N$  (see Figure 5 in [F]). In both cases the spectral condition number of the matrix  $I - R$ , which controls the propagation of errors on the data, becomes very large.

Next, we investigate the eigenvalues of the matrix  $S$  in the two channel case (see (3.8)) and of its submatrices  $S_{11}$  and  $S_{22}$ . Since the latter are real symmetric matrices, their eigenvalues give their condition number. We shall denote by  $[a]$  the maximum integer less than or equal to  $a$ . Given a  $N \times N$  matrix, we shall denote with  $\lambda_j(A)$ ,  $j = 1, \dots, N$  its eigenvalues and by  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  its minimum and maximum eigenvalue.

Let  $\mathcal{U} = \{l_1, l_2, \dots, l_N\} \subset \mathbb{Z}$  and let  $\{f(l_j t_o), f'(l_j t_o) \mid 1 \leq j \leq N\}$  a set of missing

samples. Then

$$\begin{aligned}\lambda_{\min}(S_{11}) &< 2r - r^2 < \lambda_{\max}(S_{11}) \\ \lambda_{\min}(S_{22}) &< r^2 < \lambda_{\max}(S_{22}).\end{aligned}$$

where  $r$  is the oversampling parameter (2.7). Indeed, by (3.9) and (2.10), the trace of  $S_{11}$  is  $N(2r - r^2)$ ; by (3.9) and (3.4) the trace of  $S_{22}$  is  $Nr^2$ . Moreover, since the entries of the matrix  $S$  are real, its eigenvalues are complex conjugates, thus its trace is  $N(2r - r^2) + Nr^2 = \sum_{i=1}^{2N} \operatorname{Re}(\lambda_i(S))$ ; hence

$$(4.6) \quad \operatorname{Re}(\lambda_{\min}(S)) < r < \operatorname{Re}(\lambda_{\max}(S)).$$

We observe that, if the oversampling parameter  $r$  tends to 1, i.e. the frame tends to a Riesz basis, then  $S$  tends to the identity matrix.

Following Ferreira, we shall now consider the case in which the set that locates the positions of the missing samples is  $\mathcal{U} = \{m i_1, m i_2, \dots, m i_N\}$ , where  $m$  is an integer and  $\mathcal{I} = \{i_1, i_2, \dots, i_N\}$  is a set of integers; in the following we shall denote such sets by  $m\mathcal{I}$ . The interest for studying these cases lies in the technique of interleaving the samples of a signal, prior to their transmission or archival; the advantage of this procedure is that the transmitted (or stored) information becomes less sensitive to the burst errors that typically affect contiguous set of samples ([F]). We shall investigate how the stability of the method depends on the interleaving factor  $m$ . In Proposition 4.1 and Theorem 4.2 below we find estimates for the eigenvalues of the matrices  $S_{11}, S_{22}$ , thus generalizing a result of Ferreira for one channel [F, Theorem 1].

First we consider the case  $mr$  integer. The following proposition, which extends to two channels a result in [F], shows that in this case the matrix  $S$  is lower triangular and that  $N$  eigenvalues are equal to  $2r - r^2$  and  $N$  are equal to  $r^2$ . Hence if  $r$  is rational  $r = p/q$ , one can take  $m$  equal to a multiple of  $q$  and obtain a lower-triangular matrix  $S$ .

**Proposition 4.1.** *Let  $m$  be a positive integer; suppose  $\mathcal{U} = \{m i_1, m i_2, \dots, m i_N\}$ , where  $i_j, j = 1, \dots, N$  are integers and let  $r$  be a real number in  $(0, 1)$ . If  $mr$  is an integer, then*

$$S_{11} = (2r - r^2)I \quad S_{22} = r^2 I \quad S_{12} = 0.$$

Moreover the entries of the matrix  $S_{21}$  are

$$S_{21}(k, j) = \begin{cases} (1 - r)\omega/(m\pi(i_k - i_j)) & \text{if } k \neq j \\ 0 & \text{if } k = j. \end{cases}$$

*Proof.* Since  $t_o = 2\pi r/\omega$ , by (3.9) with  $\ell_k = m i_k$ , one obtains that

$$S_{11}(k, j) = \sqrt{2\pi}\tilde{\varphi}_1\left(2\pi m \frac{r}{\omega}(i_k - i_j)\right);$$

hence by (2.10) since  $mr$  is an integer

$$S_{11}(k, j) = (2r(1 - r) + r^2)\delta_{j,k} \quad 1 \leq k \leq N, \quad 1 \leq j \leq N$$

where  $\delta_{j,k}$  is the Kronecker symbol. This shows that  $S_{11} = (2r - r^2)I_N$ . Similarly, by using (3.9) and (3.4), we obtain that  $S_{22} = r^2 I$  and  $S_{12} = 0$ . From (3.3) one finds that the off-diagonal entries of the matrix  $S_{21}$  are

$$S_{21}(k, j) = (1 - r) \frac{\omega}{m\pi(i_k - i_j)} \quad k \neq j.$$



The diagonal entries in  $S_{21}$  are equal to zero, since  $\tilde{\varphi}'_1(0) = 0$ .  $\square$

In the following theorem we find estimates for the minimum and maximum eigenvalues of the matrix  $S_{11}$  and  $S_{22}$ . The estimates do not depend on the number  $N$  of missing samples and are simple to compute.

**Theorem 4.2.** *Suppose that  $0 < r < 1$  and  $\mathcal{U} = m\mathcal{I}$ , where  $\mathcal{I} = \{i_1, i_2, \dots, i_N\}$  and  $m$  is a positive integer. If  $mr$  is not integer, then for  $j = 1, \dots, N$*

$$(4.7) \quad \alpha_{11} < \lambda_j(S_{11}) < \beta_{11}$$

$$(4.8) \quad \alpha_{22} < \lambda_j(S_{22}) < \beta_{22}$$

where

$$\begin{aligned} \alpha_{11} &= \frac{D}{m} \left(1 - \frac{D}{4m} - \frac{1}{4m}\right), & \beta_{11} &= \alpha_{11} + \frac{1}{m} \\ \alpha_{22} &= \frac{D}{4m^2} (D-1), & \beta_{22} &= \frac{D}{4m^2} (D+1) + \frac{r}{m} \end{aligned}$$

and  $D = \lfloor 2mr \rfloor$ .

*Proof.* From (2.9), by Fourier inversion and a change of variable, we obtain

$$(4.9) \quad \tilde{\varphi}_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-r}^r (1 - |y|) e^{ixy\omega/r} dy.$$

Hence by (3.9) with  $l_k = mi_k$ ,

$$S_{11}(k, j) = \int_{-r}^r (1 - |y|) e^{2\pi i m y (i_k - i_j)} dy.$$

Let  $x = (x_0, x_1, \dots, x_{N-1})$  be a column-vector in  $\mathbb{C}^N$  and denote by  $x^*$  its conjugate transpose. Then

$$\begin{aligned} x^* S_{11} x &= \sum_{k,j=0}^{N-1} \bar{x}_k S_{11}(k, j) x_j = \int_{-r}^r (1 - |y|) \sum_{k,j=0}^{N-1} \bar{x}_k x_j e^{2\pi i m y (i_k - i_j)} dy \\ &= \int_{-r}^r (1 - |y|) |P(y)|^2 dy, \end{aligned}$$

where  $P(y)$  is the  $1/m$ -periodic function

$$(4.10) \quad P(y) = \sum_{k=0}^{N-1} x_k e^{-2\pi i m y i_k}.$$

By splitting the integral in two parts and changing variable the left hand side of the above equation may be written

$$(4.11) \quad x^* S_{11} x = \int_0^r (1 - y) M(y) dy$$

where we have set

$$(4.12) \quad M(y) = |P(y)|^2 + |P(-y)|^2.$$

Next we write the integral in (4.11) as a sum of  $D+1$  integrals

$$(4.13) \quad x^* S_{11} x = \sum_{k=0}^{D-1} \int_{k/2m}^{(k+1)/2m} (1 - y) M(y) dy + \int_{D/(2m)}^r (1 - y) M(y) dy.$$

Since  $M \geq 0$  we majorize  $1 - y$  in each integral and obtain the estimate

$$(4.14) \quad x^* S_{11} x \leq \sum_{k=0}^{D-1} \left(1 - \frac{k}{2m}\right) \int_{k/2m}^{(k+1)/2m} M(y) dy + \left(1 - \frac{D}{2m}\right) \int_{D/(2m)}^r M(y) dy.$$

Next we prove that for all  $k = 0, \dots, D-1$

$$(4.15) \quad \int_{k/2m}^{(k+1)/2m} M(y) dy = \frac{1}{m}.$$

Indeed by (4.12), by changing variables and using the  $(1/m)$ -periodicity of  $P$ ,

$$\begin{aligned} \int_{k/2m}^{(k+1)/2m} M(y) dy &= \int_{k/2m}^{(k+1)/2m} |P(y)|^2 dy + \int_{-(k+1)/2m}^{-k/2m} |P(y)|^2 dy \\ &= \int_{k/2m}^{(k+1)/2m} |P(y)|^2 dy + \int_{(k+1)/2m}^{(k+2)/2m} |P(y)|^2 dy \\ &= \int_{k/2m}^{(k+2)/2m} |P(y)|^2 dy = \int_0^{1/m} |P(y)|^2 dy. \end{aligned}$$

By (4.10) and the Plancherel formula, since  $x$  has norm equal to 1, we have

$$(4.16) \quad \int_0^{1/m} |P(y)|^2 dy = \frac{1}{m}.$$

This concludes the proof of equation (4.15). Next we prove that

$$(4.17) \quad \int_{D/(2m)}^r M(y) dy < \frac{1}{m}.$$

Indeed, from (4.12), by changing variable in the integral and using the  $1/m$ -periodicity of  $P$ , we obtain

$$\begin{aligned} \int_{D/(2m)}^r M(y) dy &= \int_{D/(2m)}^r |P(y)|^2 dy + \int_{D/(2m)}^r |P(-y)|^2 dy \\ &= \int_{D/(2m)}^r |P(y)|^2 dy + \int_{D/m-r}^{D/(2m)} |P(y)|^2 dy \\ &= \int_{D/m-r}^r |P(y)|^2 dy. \end{aligned}$$

The length  $2r - D/m$  of the interval of integration is less than the period  $1/m$  of  $P$ , thus inequality (4.17) follows from (4.16).

From (4.14), by using (4.15) and (4.17), we obtain

$$(4.18) \quad x^* S_{11} x \leq \frac{1}{m} \left[ \sum_{k=0}^{D-1} \left(1 - \frac{k}{2m}\right) + \left(1 - \frac{D}{2m}\right) \right] = \frac{D}{m} \left(1 - \frac{D}{4m} - \frac{1}{4m}\right) + \frac{1}{m}$$

Thus we have proved the upper bound for the eigenvalues of  $S_{11}$ . From (4.13), by observing that the second integral is positive and using (4.15) we obtain

$$x^* S_{11} x \geq \sum_{k=0}^{D-1} \int_{k/2m}^{(k+1)/2m} (1-y) M(y) dy > \frac{1}{m} \sum_{k=0}^{D-1} \left(1 - \frac{k+1}{2m}\right) = \frac{D}{m} \left(1 - \frac{D}{4m} - \frac{1}{4m}\right).$$

This proves the lower bound and concludes the proof of inequality (4.7). The omit the proof of inequality (4.8) which is similar.  $\square$

$r$	$\alpha_{11}$	$\lambda_{\min}(S_{11})$	$\lambda_{\max}(S_{11})$	$\beta_{11}$
0.55	0.719	0.768	0.811	0.844
.6	0.773	0.813	0.859	0.898
.7	0.859	0.903	0.926	0.984
.8	0.891	0.946	0.967	1.016
.9	0.929	0.984	0.998	1.055
.95	0.936	0.996	0.999	1.063

TABLE 1.  $\lambda_{\min}(S_{11})$ ,  $\lambda_{\max}(S_{11})$  and their estimates for  $\mathcal{U} = \{0, 8, 16, 24\}$ ,  $m = 8$  and several values of  $r$ .

$r$	$\alpha_{22}$	$\lambda_{\min}(S_{22})$	$\lambda_{\max}(S_{22})$	$\beta_{22}$
.55	0.219	0.271	0.315	0.350
.6	0.281	0.317	0.391	0.427
.7	0.430	0.470	0.535	0.603
.8	0.516	0.594	0.659	0.709
.9	0.711	0.766	0.871	0.932
.95	0.820	0.877	0.962	1.056

TABLE 2.  $\lambda_{\min}(S_{22})$ ,  $\lambda_{\max}(S_{22})$  and their estimates for  $\mathcal{U} = \{0, 8, 16, 24\}$  and several values of  $r$ .

In Table 1 and Table 2 we compare the minimum and maximum eigenvalues of the matrices  $S_{11}$  and  $S_{22}$  with their estimates given by Theorem 4.2 for  $\mathcal{U} = \{0, 8, 16, 24\}$   $m = 8$  and various values of  $r$ . From Proposition 4.1 one can see that, if  $mr$  is integer, then the eigenvalues of  $S_{22}$  are smaller than the eigenvalues of  $S_{11}$ . The following corollary shows that this is also true when  $mr$  is not an integer, provided that  $m$  is sufficiently large.

**Corollary 4.3.** *Let  $m$  be a positive integer,  $\mathcal{U} = m\mathcal{I}$  and  $\mathcal{I} = \{i_1, i_2, \dots, i_N\}$ ,  $0 < r < 1$ . If  $mr$  is not an integer and  $m > (1 + 2r)/(2r(1 - r))$ , then*

$$(4.19) \quad \lambda_{\max}(S_{22}) < \lambda_{\min}(S_{11}).$$

*Proof.* By (4.7) and (4.8) it is sufficient to prove that if  $m > (1 + 2r)/(2r(1 - r))$ , then  $\beta_{22} < \alpha_{11}$ . This inequality is equivalent to  $D^2/2m + D/2m + r - D < 0$ . Since  $2mr - 1 < D \leq 2mr$ , we have

$$\frac{D^2}{2m} + \frac{D}{2m} + r - D < 2mr(r - 1) + 2r + 1$$

from which the corollary follows.  $\square$

Numerical experiments show that for  $m < (1 + 2r)/(2r(1 - r))$  the maximum eigenvalue of  $S_{22}$  may be larger than the minimum eigenvalue of  $S_{11}$ .

We shall now describe the behavior of the eigenvalues of the matrix  $S$  in dependence of the parameters  $r$  and  $N$ . The numerous experiments that we have performed suggest the conjecture that the eigenvalues of this matrix are real, positive, and less than 1. In what follows we have ordered the eigenvalues so that  $\lambda_i < \lambda_{i+1}$  for  $i = 0, \dots, N - 2$ . Figure 1 shows the largest eigenvalue of  $S$  as function of the number  $N$  of contiguous points, for various values of  $r$ . Note that, as  $r$

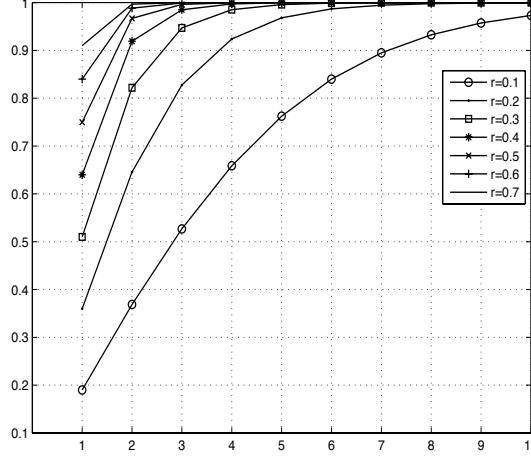


FIGURE 1. Largest eigenvalues of  $S$  as function of  $N$  for  $\mathcal{I} = \{0, 1, \dots, N-1\}$ , for various values of  $r$ .

$r$	$\text{cond}(I - S)$	$r$	$\text{cond}(I - S)$
.1	8.571 e+01	.6	2.829 e+13
.2	5.870 e+04	.7	1.661 e+16
.3	6.187 e+05	.8	2.474 e+17
.4	1.133 e+08	.9	4.096 e+17
.5	3.513 e+10	1	

TABLE 3. Condition number of  $I - S$  for  $\mathcal{U} = \{0, 1, 2, \dots, 9\}$  and  $r = 0.1, 0.2, \dots, 1$ .

approaches 1, the largest eigenvalue gets close to 1, i.e. the smallest eigenvalue of the matrix  $I - S$  tends to zero. This happens even for a small number of missing samples, as for the one-channel case (see [F], Figure 5). Thus, since the spectral condition number of a matrix is greater or equal to the ratio of the absolute values of the largest and the smallest eigenvalues,  $\text{cond}(I - S)$  becomes very large. In Table 3 we show the spectral condition numbers of  $I - S$  for  $\mathcal{U} = \{0, 1, 2, \dots, 9\}$ , for various values of  $r$ .

Figure 2 shows some of the eigenvalues of  $S$ , namely  $\lambda_i, i = 1, 5, 10, 11, 15, 20$ , as functions of the parameter  $r$  for  $\mathcal{I} = \{0, 1, 2, 3, \dots, 9\}$ ,  $m = 6$  (left) and  $m = 10$  (right). The dashed lines are the parabolas  $2r - r^2$  and  $r^2$ . One can see that for larger values of  $m$  the graphs of the eigenvalues concentrate around the graphs of the two parabolas. We observe that in the one-channel case the eigenvalues have a similar behavior (see Figure 7 in [F]).

We shall now discuss the behavior of the eigenvalues of the matrix  $S$  as the parameter  $m$  grows. By using (3.9) with  $l_k = m i_k, k = 1, \dots, N$ , (2.10), (3.3) and (3.4), one can see that, when the parameter  $m$  tends to infinity, the off-diagonal

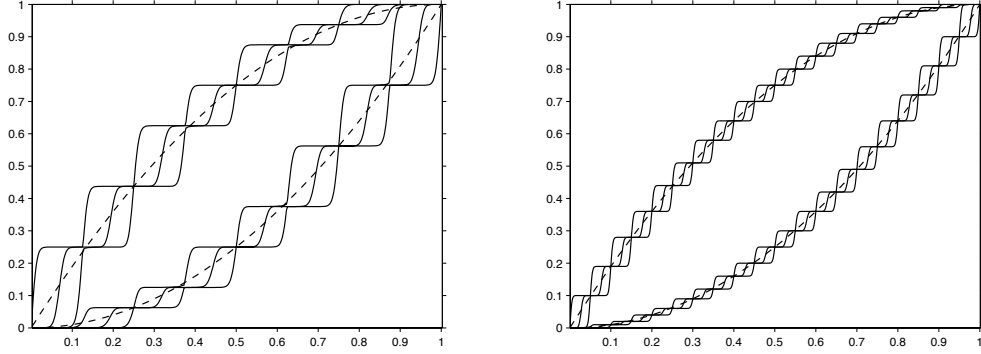


FIGURE 2. Six eigenvalues of  $S$  as functions of  $r$  for  $\mathcal{U} = \{0, 1, 2, 3, \dots, 9\}$ ,  $m = 4$  (left) and  $m = 10$  (right). Dashed lines: parabolas  $r^2$  and  $2r - r^2$

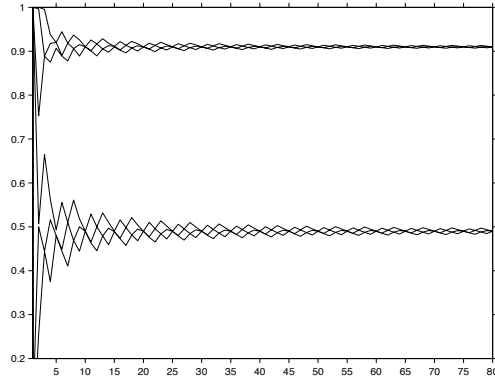


FIGURE 3. Some of the eigenvalues of  $S$  as a function of  $m$  for  $\mathcal{I} = \{0, 1, \dots, 9\}$ .  $r = .7$

entries tend to zero; for each  $\epsilon > 0$  denote by  $m_\epsilon$  the positive integer (depending on  $N$ ) such that, if  $m > m_\epsilon$ , the off-diagonal entries of the matrix are less than  $\epsilon$ . For such  $m_\epsilon$ , by the Gershgorin theorem,  $N$  eigenvalues lie in the circle of center  $2r - r^2$  and radius  $\epsilon$  and  $N$  eigenvalues lie in the circle of center  $r^2$  and radius  $\epsilon$ . Figure 3 illustrates this behavior: it shows the eigenvalues  $\lambda_i$   $i = 1, 5, 10, 11, 15, 20$  in dependence of the parameter  $m$  for  $r = .7$  and  $\mathcal{I} = \{0, 1, \dots, 9\}$ . Note that the limits of the eigenvalues are  $2r - r^2$  and  $r^2$ .

## 5. RECONSTRUCTION: NUMERICAL RESULTS

In this section we present some numerical experiments on the recovery of missing samples and on the reconstruction of a signal via oversampling formulas. In particular, we analyze the case of contiguous missing samples, when the problem may become very ill-conditioned, depending on the parameter  $r$  and the number of missing samples. Indeed if  $r$  is close to 1, or the number of missing samples is large, the condition number of the matrix  $I - S$  is large, so that the system amplifies

the errors in the data. We solve this problem, also in presence of noisy data, by applying a regularization technique typical of the treatment of inverse problems (see [BB], Section 5). These techniques consist in considering a family of approximate solutions  $X_\lambda$  depending on a non-negative parameter  $\lambda$ , called *regularization parameter*. When the data are noise-free, the solution  $X_\lambda$  converges to the exact solution as the regularization parameter tends to zero. In the case of noisy data, one can obtain an optimal approximation of the exact solution for a positive value of the parameter.

Following Ferreira, in all our experiments we shall use the test function

$$(5.1) \quad g(x) = (\text{sinc}(\pi(x - 2.1))) - 0.7(\text{sinc}(\pi(x + 1.7))).$$

which has band  $[-\pi, \pi]$  [F].

First we present some numerical experiments for the recovery of contiguous missing samples via the one-channel formula (4.1). To compute the sum in (4.5), we must truncate it to the values  $k$  such that  $|k| \leq M$ , for some integer  $M$ , thus introducing an error. In [F] the author makes the choice  $M = 40$  to reconstruct the signal (5.1) when  $\mathcal{U} = \{0, 1, \dots, 5\}$  and  $r = 0.6$  (hence  $t_o = 0.6$ ). One can verify that for  $M = 40$  the norm of the truncation error is of order  $10^{-3}$  and that choosing  $M = 500$  this error is reduced to the order of  $10^{-4}$ , while, due to round-off errors, a larger value of  $M$  does not reduce it further. In Table 4 we show the values  $x_k = l_k t_o$  (column 1) the exact samples  $g(x_k)$  (column 2), the computed samples  $g_k^*$  and the relative errors  $e_k^*$  with  $M = 40$  (columns 3 and 4), the computed samples  $g_k$  and the relative errors  $e_k$  with  $M = 500$  (columns 5 and 6). The improved results obtained in the latter case can be explained by observing that the condition number of the matrix  $I - R$ , which controls the propagation of the errors, is  $3.07 \cdot 10^4$ . We have

$x_k$	$g(x_k)$	$g_k^*$	$e_k^*$	$g_k$	$e_k$
0.0	0.1529	0.1132	0.2598	0.1498	0.0199
0.6	-0.2906	-0.5344	0.8390	-0.3096	0.0653
1.2	0.0856	-0.4833	6.6498	0.0410	0.5206
1.8	0.9221	0.2132	0.7687	0.8664	0.0603
2.4	0.8416	0.3498	0.5844	0.8029	0.0459
3.0	0.0710	-0.0872	2.2288	0.0585	0.1751

TABLE 4. Recovered samples with the one-channel formula,  $r=0.6$ : exact samples (col 2), computed samples and relative errors with  $M = 40$  (col.s 3 and 4) and  $M = 500$  (col.s 5 and 6).

also plotted the graphs: in Figure 4 we compare the original and the reconstructed signal in both cases. For  $M = 500$  we have zoomed the graphs, representing them in the interval  $[-5, 5]$ , to make them distinguishable. By the preceding considerations, in all the experiments described below, we have chosen  $M = 500$  to reduce the error due to the truncation error.

Next we consider the problem of data affected by noise; we choose again  $r = 0.6$  and  $\mathcal{U} = \{-2, -1, 0, 1, 2, 3\}$ . The condition number of the matrix  $I - R$  is still  $3.0769 \cdot 10^4$ , since it does not depend on the position of the contiguous missing samples (see (4.4)). We have introduced a random noise of order  $10^{-2}$  on the data

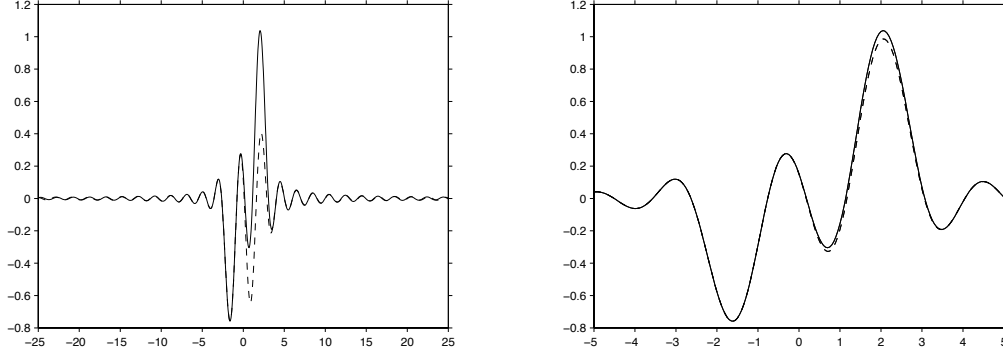


FIGURE 4. Original (solid line) and reconstructed (dotted line) signal  $g$ ,  $r = .6$ ,  $\mathcal{U} = \{0, 1, \dots, 5\}$ . Truncation  $M = 40$  (left) and  $M = 500$  (right), on the interval  $[-5, 5]$ .

and have solved the system (3.10) by using the Tikhonov regularization technique. Since an estimate of the norm of the noise is known, it is possible to use the *discrepancy principle* to choose the value of the regularization parameter  $\lambda$  (see [BB] and [M]). Table 5 summarizes the results of this experiment; columns 1 and 2 contain the values  $x_k = l_k t_o$  and the exact values  $g(x_k)$  of the missing samples,

$x_k$	$g(x_k)$	$g_k$	$e_k$	$g_k^r$	$e_k^r$
-1.2	-0.5237	50.9126	98.2227	-0.4852	0.0734
-0.6	0.1580	180.6426	1142.5116	0.2141	0.3550
0.0	0.1529	306.9469	2006.8097	0.1430	0.0652
0.6	-0.2906	307.9712	1060.8440	-0.3608	0.2416
1.2	0.0856	183.2893	2141.4934	0.0435	0.4918
1.8	0.9221	53.8130	57.3621	0.9248	0.0030

TABLE 5. Recovered samples with the one-channel formula,  $r=0.6$ , with noise added: exact samples (col 2), computed samples and relative errors without regularization (col.s 3 and 4), computed samples with regularization and the relative errors (col.s 5 and 6).

columns 3 and 4 contain the recovered samples  $g_k$  and the relative errors  $e_k$  obtained without regularization, while columns 5 and 6 show the recovered samples  $g_k^r$  and the relative errors  $e_k^r$  obtained via regularization. As in the previous example by comparing columns 2 and 5, one can see that the absolute errors are very small: the plots of the real and reconstructed signal would be indistinguishable.

We shall now present some experiments that involve the two-channel formula. To compute the right-hand side of (3.10), we have truncated the sums in (3.6) retaining only the terms for which  $|n| \leq 500$ . As in the one-channel case, this leads to an error of order  $10^{-4}$ . Our experiments show that, as in the one-channel case, the recovery and the reconstruction of missing samples of the function and of its derivative is rather efficient if the distance between the missing samples is large

and the oversampling parameter  $r$  is not too close to 1. To give an example, for  $\mathcal{I} = \{-1, 0, 1, 2, 3, 4\}$ ,  $m = 4$  and  $r = 0.7$ , the maximum absolute error is less than  $8 \cdot 10^{-4}$ . The error increases when the parameter  $m$  gets smaller, or  $r$  larger.

Next, we experiment the recovery of consecutive missing samples of  $g$  and  $g'$  when  $\mathcal{U} = \{-2, -1, 0, 1, 2, 3\}$ ,  $r = 0.6$  (hence  $t_o = 1.2$ ). The results are shown in Tables 6 and 7. In this case the condition number of the matrix  $I - S$  in (3.10) is  $3.67 \cdot 10^7$ , much larger than in the one-channel case. This has the effect of propagating strongly the truncation error, thus producing huge errors of the solution, much worse than in the one-channel case. Thus, with this distribution of missing points and this value of the parameter  $r$  the system is very ill-conditioned; an application of the Tikhonov regularization technique with discrepancy principle leads to the results in columns 5 and 6 in Table 6 and 7. In Figure 5 we show the plots of the original and the computed signal  $g$ .

$x_k$	$g(x_k)$	$g_k$	$e_k$	$g_k^r$	$e_k^r$
-2.4	-0.1868	-0.0580	-0.6893	-0.0785	0.5799
-1.2	-0.5237	2.1433	-5.0929	-0.4569	0.1274
0.0	0.1528	10.3983	67.0174	0.1832	0.1989
1.2	0.0856	12.0422	139.7624	-0.1897	3.2170
2.4	0.8416	5.1810	5.1561	-0.1480	1.1758
3.6	-0.1782	0.1425	-1.7997	-0.4405	1.4720

TABLE 6. Recovered samples of  $g$  with two channels,  $r = 0.6$ , no noise added: exact samples (col. 2), computed samples and relative errors (cols 3 and 4), computed samples and the relative errors (cols 5 and 6) with regularization.

$x_k$	$g'(x_k)$	$g'_k$	$e'_k$	$g'_k{}^r$	$e'_k{}^r$
-2.4	-0.9399	-0.4742	-0.4955	-0.7919	0.1575
-1.2	1.0457	5.5797	4.3356	0.8325	0.2039
0.0	-0.7350	5.4046	-8.3534	-0.5690	0.2258
1.2	1.4159	-2.6352	2.8611	0.5867	0.5857
2.4	-1.0603	-7.1265	-5.7212	-0.9327	0.1203
3.6	0.2127	-0.8092	4.8046	0.8036	2.7786

TABLE 7. Recovered samples of  $g'$  with two channels,  $r = 0.6$ , no noise added: exact samples (col. 2), computed samples and relative errors (cols 3 and 4), computed samples and the relative errors (cols 5 and 6) with regularization.

Of course, by using a smaller oversampling parameter, one can reduce the condition number of the matrix  $I - S$ , obtaining much better results: using  $r = 0.3$  (hence  $t_o = 0.6$ ), and the same set  $\mathcal{U} = \{-2, -1, 0, 1, 2, 3\}$ , one gets  $\text{cond}(I - S) = 1.92 \cdot 10^4$  and maximum absolute error on the samples of the signal equal to 0.0078. In Section 2 we have observed that, in the two-channel formula, the case  $r \in (0, .5)$  implies oversampling in each channel separately, thus it can be used the one-channel formula separately for the signal and its derivative. However, the two alternative



are not numerically equivalent. To recover the same missing samples via the one-channel formula, we have taken  $r = 0.6$  and  $\mathcal{U} = \{-2, -1, 0, 1, 2, 3\}$  and we have obtained a maximum absolute error on the samples of the signal equal to 0.0274. This shows that the second alternative may be less efficient in case of consecutive samples.

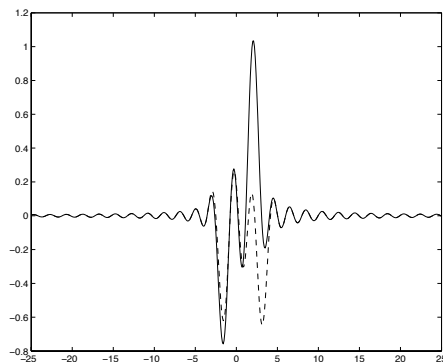


FIGURE 5. Original and reconstructed signal  $g$ , recover with regularization, for  $\mathcal{U} = \{-2, -1, 0, 1, 2, 3\}$   $r = 0.6$ .

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